Math 275D Lecture 23 Notes

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November 25, 2019

1 Constructing Martingales and Itô Integration in $\mathbb{R}^d$

1.1 Constructing martingales using Itô integration

If we have $f(t, B_t)$, then Itô's formula gives us

$$df(t, B_t) = f_t\,dt + f_x dB_s + \frac{1}{2} f_{xx}\,dt$$

If $f_t = -\frac{1}{2} f_{xx}$, we get that

$$f(t, B_t) = \int f_x\,dB_s.$$

This implies that $f(t, B_t)$ is a local martingale. If this is bounded, it must be a martingale.

Example 1.1. Suppose we have the line $\mu t + a$. What is $\mathbb{P}(\exists t \text{ s.t. } B_t = \mu t + a)$? Let's try looking at $X_t = B_t - \mu t$. Then we want $\mathbb{P}(\exists t \text{ s.t. } X_t = a)$.

In general, let's look at $X_t = \mu t + \sigma B_t$, where $\mu, \sigma \in \mathbb{R}$. If we let $\tau = \inf\{t : X_t = a \text{ or } b\}$, then the probability we want is the limit $\lim_{b \to \infty} \mathbb{P}(X_\tau = a)$. If $\mu = 0$, then X_t is a martingale. Then $Y_t := \frac{X_t - b}{a - b}$ is also a martingale, and

$$Y_\tau = \begin{cases} 1 & X_\tau = a \\ 0 & X_\tau = b. \end{cases}$$

Then

$$\mathbb{P}(Y_\tau = 1) = \mathbb{E}[Y_\tau] = \mathbb{E}[Y_0] = -\frac{b}{a - b}.$$

The middle step comes from the fact that $Y_{t \wedge \tau}$ is bounded.

Example 1.2. Let's find some function with the form $f(t, x) = \widetilde{f}(\mu t + \sigma z)$ and $f_t + \frac{1}{2} f_{xx} = 0$; such a function will make $\widetilde{f}(X_t)$ a local martingale. The derivative condition is $\mu \widetilde{f}' + \frac{1}{2}\sigma^2 \widetilde{f}'' = 0$. Notice that if $\mu = 0$, then $\widetilde{f}(s) = cs + d$ for some c, d; so $cX_t + d$ is a martingale.

Let's try the function

$$\widetilde{f}_{A,B}(s) = \frac{e^{-2\mu s/\sigma^2} - e^{-2\mu B/\sigma^2}}{e^{-2\mu A/\sigma} - e^{-2\mu B/\sigma^2}}.$$

Then $\widetilde{f}$ satisfies this differential equation, $\widetilde{f} = 1$ if $s = A$, and $\widetilde{f} = 0$ if $s = B$. So

$$\mathbb{E}[\widetilde{f}_{A,B}(X_t)] = \mathbb{P}(X_\tau = A) = \mathbb{E}[\widetilde{f}_{A,B}(X_0)].$$

1.2 Brownian motion and Itô integration in $\mathbb{R}^d$

We can construct Brownian motion in $\mathbb{R}^d$ by constructing a length d vector of independent Brownian motions $B(t) = (B_1(t), B_2(t), \dots, B_d(t))$. Here is what Itô's formula looks like in $\mathbb{R}^d$:

Theorem 1.1. *For suitable $f(t, B(t))$.*

$$\begin{aligned} df(t, B(t)) &= f_t\,dt + \sum_{k=1}^d f_{x_k}\,dB_k + \frac{1}{2}\sum_{k=1}^d f_{x_k x_k}\,dt \\ &= f_t\,dt + \nabla f \cdot dB + \frac{1}{2}\Delta f\,dt. \end{aligned}$$

Equivalently,

$$f(t, B(t)) = \int_0^t f_t(s, B(s))\,ds + \sum_{k=1}^d f_{x_k}(s, B(s))\,dB_k(s) + \frac{1}{2}\int_0^t f_{x_k x_k}(s, B(s))\,dt.$$

Corollary 1.1. *If $f(t,x)$ satisfies $f_t + \frac{1}{2}\Delta f = 0$, then $f(t, B_t)$ is a local martingale.*

Look at 2-dimensional brownian motion. Then for $a \in \mathbb{R}^2$,

$$\mathbb{P}(\exists t \text{ s.t. } B_t = a) = 0.$$

We can also find the following.

Proposition 1.1.

$$\mathbb{P}(\exists \{t_k\} \textit{ s.t. } \lim_k B_{t_k} = a) = 1.$$

This is a nontrivial question to ask. However, this is a well-known result nowadays. Interestingly, this probability is not equal to 1 when $d > 2$.

Proof. We want to find $\mathbb{P}_a(\exists t \text{ s.t. } |B_t| \le r)$; if this probability is 1 for $r > 0$, then the result holds by taking a sequence of r going to 0. Let $\tau := \inf\{t : |B_t| = r \text{ or } r_L\}$, where $r_L > R$. Then we want $\mathbb{P}(|B_\tau| = r)$. Let's find some $f(t, B_t)$ such that $f_t + \frac{1}{2}f_{xx} = 0$. We also need a kind of property like $f(t,x)=1$ if $|B_t|=r$ and $f(t,B_t)=0$ if $|B_t|=r_L$. We can do this by requiring $f(t,x)=h(\|x\|)$, i.e. it only depends on h. This gives $\Delta f=0$. So we want a function like $h(x)\log|x|$. The actual choice is

$$f(x)=\frac{\log|x|-\log(r_L)}{\log(r)-\log(r_L)}.$$

This gives

$$\mathbb{P}(|B_\tau|=r)=\mathbb{E}[f(B(t))]=\mathbb{E}[f(B(0))]=\frac{\log(R)-\log(r_L)}{\log(r)-\log(r_L)}\xrightarrow{r_L\to\infty}1.$$

This completes the proof. □

Remark 1.1. If $d=3$, we get that $f(x)$ is like $1/|x|$. In particular,

$$f(x)=\frac{1/|x|-1/|r_L|}{1/|r|-1/|r_L|}.$$

The same calculation gives

$$\mathbb{P}(|B_\tau|=r)=\frac{1/R-1/r_L}{1/r-1/r_L}\xrightarrow{r_L\to\infty}\frac{r}{R}.$$